Game Theory, Spring 2024

Lecture # 11

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1 Correlated equilibria

Example 1. *Consider the following strategic-form game:*

	L	R
T	$4,4$	$1,5$
B	$5,1$	$0,0$

The game in Example 1 has three Nash equilibria: two equilibria in pure strategies: (B, L) and (T, R); and one equilibrium in mixed strategies: $\left(\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R\right)$.

We can represent these equilibria as joint distributions over the realized action profiles. Let $\alpha(a)$ denote the probability that the action profile $a \in A$ is palyed, i.e.

	L	R
T	$\alpha(T, L)$	$\alpha(T, R)$
B	$\alpha(B, L)$	$\alpha(B, R)$

- (B, L) can then be represented as:

	L	R
T	0	0
B	1	0

- (T, R) can then be represented as:

	L	R
T	0	1
B	0	0

- $\left(\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R\right)$ can then be represented as:

	L	R
T	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{4}$	$\frac{1}{4}$

Nash equilibrium allows players to mix across their actions independently. We are now going to generalize it by allowing them to play correlated strategies. To motivate this generalization, suppose there is a mediator who issues recommendations for the players. More precisely, the mediator draws an action profile $(a_1, \ldots, a_I)$ according to some distribution α over the set of all action profiles A, and then privately communicates her recommnedation a_i to each player i. We assume that the mediator is not interesed in the outcome of the game, and thus can commit to any distribution $\alpha \in \Delta(A)$. To define an equilibrium, we need to make sure that α is chosen in a way that incentivizes the players to follow the recommendations. In Example 1, this amounts to checking whether α satisfies the following incentive constraints:

- if player 1 is recommended to play T:

$$(\text{IC}_T) \qquad \underbrace{\alpha(L|T)4 + \alpha(R|T)1}_{\text{play } T} \geq \underbrace{\alpha(L|T)5 + \alpha(R|T)0}_{\text{play } B \text{ instead}},$$

- if player 1 is recommended to play B:

$$(\text{IC}_B) \qquad \underbrace{\alpha(L|B)5 + \alpha(R|B)0}_{\text{play } B} \geq \underbrace{\alpha(L|B)4 + \alpha(R|B)1}_{\text{play } T \text{ instead}},$$

- if player 2 is recommended to play L:

$$(\text{IC}_L) \qquad \underbrace{\alpha(T|L)4 + \alpha(B|L)1}_{\text{play } L} \geq \underbrace{\alpha(T|L)5 + \alpha(B|L)0}_{\text{play } R \text{ instead}},$$

- if player 2 is recommended to play R:

$$(\text{IC}_R) \qquad \underbrace{\alpha(T|R)5 + \alpha(B|R)0}_{\text{play } R} \geq \underbrace{\alpha(T|R)4 + \alpha(B|R)1}_{\text{play } L \text{ instead}}.$$

If α satisfies the incentive constraints, we are going to call α a *correlated equilibrium.* Here is the general definition:

Definition 1 (**Correlated equilibrium**). *A correlated equilibrium is a probability distribution over action profiles $\alpha \in \Delta(A) = \Delta(A_1 \times \cdots \times A_I)$ such that for every player i and every action $a_i \in A_i$ recommended with positive probability (i.e. such that $\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i}) > 0$), and every action $\tilde{a}_i \in A_i$ the following incentive constraint is satisfied:*

$$\sum_{a_{-i} \in A_{-i}} \alpha(a_{-i}|a_i) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \alpha(a_{-i}|a_i) u_i(\tilde{a}_i, a_{-i}).$$

We can rewrite the incentive constraint in Definition 1 as follows:

$$\begin{aligned}
&\sum_{a_{-i} \in A_{-i}} \frac{\alpha(a_i, a_{-i})}{\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i})} u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \frac{\alpha(a_i, a_{-i})}{\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i})} u_i(\tilde{a}_i, a_{-i}) \\
\Leftrightarrow &\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i}) u_i(\tilde{a}_i, a_{-i}) \\
\Leftrightarrow &\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i}) \big[u_i(a_i, a_{-i}) - u_i(\tilde{a}_i, a_{-i})\big] \geq 0. \qquad (1)
\end{aligned}$$

Observe that Inequality (1) is trivially satisfied if $\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i}) = 0$ (i.e. when a_i is recommended with probability 0), hence we can equivalently define correlated equilibria as follows:

Definition 2 (**Correlated equilibrium**). *A correlated equilibrium is a probability distribution over action profiles $\alpha \in \Delta(A) = \Delta(A_1 \times \cdots \times A_I)$ such that for every player i, every action $a_i \in A_i$ and every action $\tilde{a}_i \in A_i$ the following incentive constraint is satisfied:*

$$\sum_{a_{-i} \in A_{-i}} \alpha(a_i, a_{-i}) \big[u_i(a_i, a_{-i}) - u_i(\tilde{a}_i, a_{-i})\big] \geq 0.$$

The advantage of Definition 2 comes from the fact that the incentive constraints are given by a system of linear inequalities.

Let us go back to Example 1, and introduce more convenient notation for the probabilities of action profiles:

	L	R
T	$\alpha(T, L) = x$	$\alpha(T, R) = y$
B	$\alpha(B, L) = z$	$\alpha(B, R) = w$

Rewriting the incentive constraints as in Definition 2, we get the following system of correlated equilibrium conditions:

$$
\begin{aligned}
&(\mathrm{IC}_T) && -x + y \geq 0,\\
&(\mathrm{IC}_B) && z - w \geq 0,\\
&(\mathrm{IC}_L) && -x + z \geq 0,\\
&(\mathrm{IC}_R) && y - w \geq 0,\\
&(\mathrm{Prob}) && x + y + z + w = 1,\\
&(\mathrm{NN}) && x \geq 0,\ y \geq 0,\ z \geq 0,\ w \geq 0.
\end{aligned}
$$

It is not hard to verify that all the Nash equilibria in Example 1 satisfy this system, and thus are correlated equilibria. Indeed, this fact is true more generally:

Fact 1. *Any Nash equilibrium is a correlated equilibrium.*

We can now ask whether there are correlated equilibria that are not Nash equilibria. The following claim establishes that the answer to this question is yes.

Claim 1. *The following distribution is a correlated equilibrium in Example 1.*

	L	R
T	$\frac{1}{3}$	$\frac{1}{3}$
B	$\frac{1}{3}$	0

Proof.

$$
\begin{aligned}
&(\mathrm{IC}_T) \qquad && -x+y=-\frac{1}{3}+\frac{1}{3}=0,\\
&(\mathrm{IC}_B) \qquad && z-w=\frac{1}{3}-0=\frac{1}{3}\geq 0,\\
&(\mathrm{IC}_L) \qquad && -x+z=-\frac{1}{3}+\frac{1}{3}=0,\\
&(\mathrm{IC}_R) \qquad && y-w=\frac{1}{3}-0=\frac{1}{3}\geq 0.
\end{aligned}
$$

□

We would like to characterize the whole set of correlated equilibria. To gain some intuition, let us look at the subset of correlated equilibria in Example 1, for which $w=0$, i.e. at the correlated equilibria of the following form:

	L	R
T	x	y
B	z	0

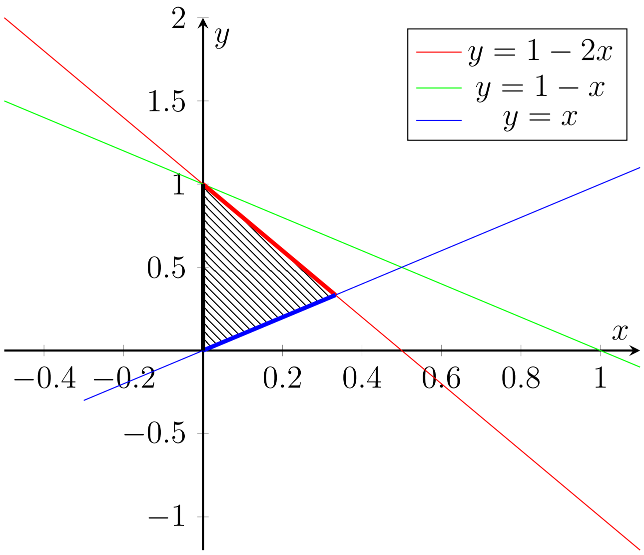
The equilibrium conditions in this subset can be written as follows:

$$
\begin{aligned}
&(\mathrm{IC}_T) \qquad && -x+y\geq 0,\\
&(\mathrm{IC}_B) \qquad && z\geq 0,\\
&(\mathrm{IC}_L) \qquad && -x+z\geq 0,\\
&(\mathrm{IC}_R) \qquad && y\geq 0,\\
&(\mathrm{Prob}) \qquad && x+y+z=1,\\
&(\mathrm{NN}) \qquad && x\geq 0,\ y\geq 0,\ z\geq 0.
\end{aligned}
$$

Observe that (IC_B) and (IC_R) now become redundant. Solving for $z=1-x-y$, we get the following system of inequalities:

$$
\begin{cases}
y\geq x,\\
y\leq 1-2x,\\
y\leq 1-x\\
x\geq 0,\ y\geq 0.
\end{cases}
\tag{2}
$$

The solutions to the system of inequalities (2) can be found graphically.



Observe that this subset of correlated equilibria of Example 1 is a convex and compact set. These properties hold for the whole set of correlated equilibria of any strategic-form game.

2 Properties of the set of correlated equilbria

Let $\Gamma = \big(\mathcal{I}, A, \{u_i\}_{i\in\mathcal{I}}\big)$ be a strategic form game, and let $CE(\Gamma)$ be the set of its correlated equilibria. We establish the following propositions:

Proposition 1 (Convexity). *$CE(\Gamma)$ is a convex set.*

Proof. Suppose $\alpha' \in CE(\Gamma)$ and $\alpha'' \in CE(\Gamma)$, and consider $\alpha = \lambda\alpha' + (1-\lambda)\alpha''$ for some $\lambda \in (0,1)$. We show that $\alpha \in CE(\Gamma)$ as well. First of all, α is clearly a probability distribution since $\alpha(a) = \lambda\alpha'(a) + (1-\lambda)\alpha''(a) \geq 0$ for any $a \in A$, and

$$\sum_{a\in A} \alpha(a) = \sum_{a\in A} \Big[\lambda\alpha'(a) + (1-\lambda)\alpha''(a)\Big] = \lambda \underbrace{\sum_{a\in A} \alpha'(a)}_{=1} + (1-\lambda) \underbrace{\sum_{a\in A} \alpha''(a)}_{=1} = 1.$$

Moreover α satisfies the incentive constraints: take any $a_i, \tilde{a}_i \in A_i$ and consider

$$\begin{aligned}
&\sum_{a_{-i}\in A_{-i}} \alpha(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big] \\
&= \sum_{a_{-i}\in A_{-i}} \Big[\lambda\alpha'(a_i,a_{-i}) + (1-\lambda)\alpha''(a_i,a_{-i})\Big]\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big] \\
&= \lambda \underbrace{\sum_{a_{-i}\in A_{-i}} \alpha'(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big]}_{\geq 0 \text{ by IC for } \alpha'} + (1-\lambda) \underbrace{\sum_{a_{-i}\in A_{-i}} \alpha''(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big]}_{\geq 0 \text{ by IC for } \alpha''} \geq 0.
\end{aligned}$$

□

Corollary 1. *Any convex combination of Nash equilibria is a correlated equilibrium.*

Proposition 2 (Compactness). *$CE(\Gamma)$ is a compact set.*

Proof. $CE(\Gamma)$ is bounded since $CE(\Gamma) \subseteq \Delta(A)$, which itslelf is clearly bounded. To show closedness, consider a convergent sequence of correlated equilbria $\{\alpha^n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} \alpha^n = \alpha$. We show $\alpha \in CE(\Gamma)$. First of all, α is a probability distribution since $\alpha(a) = \lim_{n\to\infty\alpha^n(a)} \geq 0$ since $\alpha^n(a) \geq 0$ for every $a \in A$ and $\sum_{a\in A} \alpha(a) = \lim_{n\to\infty} \sum_{a\in A} \alpha^n(a) = 1$.

Moreover, α satisfies the incentive constraints: take any $a_i, \tilde{a}_i \in A_i$ and observe

$$\sum_{a_{-i}\in A_{-i}} \alpha(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big] = \lim_{n\to\infty} \underbrace{\sum_{a_{-i}\in A_{-i}} \alpha^n(a_i,a_{-i})\big[u_i(a_i,a_{-i}) - u_i(\tilde{a}_i,a_{-i})\big]}_{\geq 0 \text{ by IC for } \alpha^n \text{ for each } n} \geq 0.$$

□