Game Theory, Spring 2024

Lecture # 9

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1 The one-shot deviation principle

At the end of Lecture #8 we stated the following proposition[1]:

Proposition 1 (The one-shot deviation principle). *A strategy profile σ is a subgame-perfect Nash equilibrium iff there are no profitable one-shot deviations.*

Proof. Clearly, if σ is a subgame-perfect Nash equilibrium, then there are no profitable one-shot devitations. Suppose now that σ is not a subgame-perfect Nash equilibrium, we will show that there exists a profitable one-shot deviation. If σ is not a subgame-perfect Nash equilibrium, then there exists history $\tilde{h}^t$ and a strategy $\tilde{\sigma}_i$ for player i such that

$$U_i\big(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big) < U_i\big(\tilde{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big).$$

We first establish the following lemma:

Lemma 1. *There is a period T, sufficiently distant in the future, such that*

$$\hat{\sigma}_i(h^\tau) \equiv \begin{cases} \tilde{\sigma}_i(h^\tau) & \text{if } \tau < T, \\ \sigma_i|_{\tilde{h}^t}(h^\tau) & \text{if } \tau \geq T, \end{cases}$$

[1] The proof of the one-shot deviation principle is adapted from *"Repeated Games and Reputations"* by George J. Mailath and Larry Samuelson.

is also a profitable deviation from $\sigma_i|_{\tilde{h}^t}$*, i.e.*

$$U_i\big(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big) < U_i\big(\hat{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big).$$

Proof. Denote $m \equiv \min_{i,a} u_i(a)$ and $M \equiv \max_{i,a} u_i(a)$, and also $a^\tau \equiv a^\tau\big[\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big]$, $\tilde{a}^\tau \equiv a^\tau\big[\tilde{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big]$ and $\hat{a}^\tau \equiv a^\tau\big[\hat{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big]$; and observe that

$$\begin{aligned} U_i\big(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big) &= (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(a^\tau) + (1-\delta)\sum_{\tau=T}^{\infty}\delta^\tau u_i(a^\tau) \\ &\geq (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(a^\tau) + \delta^T m, \end{aligned}$$

and

$$\begin{aligned} U_i\big(\tilde{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big) &= (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(\tilde{a}^\tau) + (1-\delta)\sum_{\tau=T}^{\infty}\delta^\tau u_i(\tilde{a}^\tau) \\ &\leq (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(\tilde{a}^\tau) + \delta^T M. \end{aligned}$$

Denoting $\epsilon \equiv U_i\big(\tilde{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big) - U_i\big(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big) > 0$, we obtain:

$$\epsilon - \delta^T(M-m) \leq (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(\tilde{a}^\tau) - (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(a^\tau).$$

As long as T is far away enough from t, we get $\frac{\epsilon}{2} < \epsilon - \delta^T(M-m)$, and therefore

$$(1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(\tilde{a}^\tau) - (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(a^\tau) - \frac{\epsilon}{2} > 0. \tag{1}$$

Observe now that

$$\begin{aligned} U_i\big(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big) &= (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(a^\tau) + (1-\delta)\sum_{\tau=T}^{\infty}\delta^\tau u_i(a^\tau) \\ &\leq (1-\delta)\sum_{\tau=0}^{T-1}\delta^\tau u_i(a^\tau) + \delta^T M, \end{aligned}$$

and

$$
\begin{aligned}
U_i\big(\hat{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big) &= (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(\tilde{a}^\tau) + (1-\delta)\sum_{\tau=T}^{\infty} \delta^\tau u_i(\hat{a}^\tau) \\
&\geq (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(\tilde{a}^\tau) + \delta^T m,
\end{aligned}
$$

Let $\hat{\epsilon} \equiv U_i\big(\hat{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big) - U_i\big(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t}\big)$, we then have:

$$
\begin{aligned}
\hat{\epsilon} &\geq (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(\tilde{a}^\tau) + \delta^T m - (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(a^\tau) - \delta^T M \\
&= (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(\tilde{a}^\tau) - (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(a^\tau) - \delta^T (M-m) \\
&= \underbrace{(1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(\tilde{a}^\tau) - (1-\delta)\sum_{\tau=0}^{T-1} \delta^\tau u_i(a^\tau) - \frac{\epsilon}{2}}_{>0 \text{ by Inequality (1)}} + \underbrace{\left(\frac{\epsilon}{2} - \delta^T(M-m)\right)}_{>0 \text{ as } \frac{\epsilon}{2} < \epsilon - \delta^T(M-m)} > 0.
\end{aligned}
$$

□

The rest of the proof of the one-shot deviation principle is by backward induction on the value of T. Consider period $T-1$ first. Let $\hat{h}^{T-1} \equiv \big(\hat{a}^0, \ldots, \hat{a}^{T-1}\big)$ denote the $T-1$-period history induced by playing $\big(\hat{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}\big)$. We distinguish two cases:

1. $U_i\big(\sigma_i|_{\tilde{h}^t\hat{h}^{T-1}}, \sigma_{-i}|_{\tilde{h}^t\hat{h}^{T-1}}\big) < U_i\big(\hat{\sigma}_i|_{\hat{h}^{T-1}}, \sigma_{-i}|_{\tilde{h}^t\hat{h}^{T-1}}\big)$. In this case, we are done as $\hat{\sigma}_i|_{\hat{h}^{T-1}}$ is the desired one-shot deviation from $\sigma_i|_{\tilde{h}^t\hat{h}^{T-1}}$.
2. $U_i\big(\sigma_i|_{\tilde{h}^t\hat{h}^{T-1}}, \sigma_{-i}|_{\tilde{h}^t\hat{h}^{T-1}}\big) \geq U_i\big(\hat{\sigma}_i|_{\hat{h}^{T-1}}, \sigma_{-i}|_{\tilde{h}^t\hat{h}^{T-1}}\big)$. In this case, define

$$
\bar{\sigma}_i(h^\tau) \equiv \begin{cases} \hat{\sigma}_i(h^\tau) & \text{if } \tau < T-1, \\ \sigma_i|_{\tilde{h}^t}(h^\tau) & \text{if } \tau \geq T-1. \end{cases}
$$

The following lemma applies:

Lemma 2. *$\bar{\sigma}_i$ is a profitable deviation from $\sigma_i|_{\tilde{h}^t}$.*

Proof. Consider the payoff of player i from $(\bar{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t})$:

$$\begin{aligned} U_i(\bar{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}) &= (1-\delta)\sum_{\tau=0}^{T-2} \delta^\tau u_i(\hat{a}^\tau) + \delta U_i(\sigma_i|_{\tilde{h}^t \hat{h}^{T-1}}, \sigma_{-i}|_{\tilde{h}^t \hat{h}^{T-1}}) \\ &\geq (1-\delta)\sum_{\tau=0}^{T-2} \delta^\tau u_i(\hat{a}^\tau) + \delta U_i(\hat{\sigma}_i|_{\hat{h}^{T-1}}, \sigma_{-i}|_{\tilde{h}^t \hat{h}^{T-1}}) \\ &= \underbrace{U_i(\hat{\sigma}_i, \sigma_{-i}|_{\tilde{h}^t}) > U_i(\sigma_i|_{\tilde{h}^t}, \sigma_{-i}|_{\tilde{h}^t})}_{\text{by Lemma 1}}. \end{aligned}$$

□

Repeating the above argument for periods from $T-2$ to the initial period, we will find a profitable one-shot deviation.

□

2 Other equilibria of repeated prisoner's dilemma

Let us go back to Example 4 from Lecture #8:

Example 4. *Consider the infinite repetition of the following prisoner's dilemma:*

	c	d
c	$5,5$	$1,6$
d	$6,1$	$2,2$

Consider the following two-phase strategy σ^* (k-punishment strategy):

- Start in the *regular* phase. In the regular phase, play c.
- If anyone has played d in the regular phase, switch to the *punishment phase*, otherwise stay in the regular phase.
- In the punishment phase, play d for k periods, then go back to the regular phase.

We establish the following claim:

Proposition 2. (σ^*, σ^*) *is a subgame-perfect Nash equilibrium of the repeated prisoner's dilemma in Example 4 for all sufficiently high values of* δ.

Proof. The proof is by the one-shot deviation principle. Consider any history in the regular phase. If a player sticks to her equilibrium strategy, her payoff is:

$$(1-\delta)\big(5+\delta 5+\delta^2 5+\delta^3 5+\dots\big) = (1-\delta)\sum_{t=0}^{\infty}\delta^t 5 = (1-\delta)\frac{5}{1-\delta} = 5.$$

If a player attempts a one-shot deviation in the regular phase, her payoff becomes:

$$\begin{aligned}
&(1-\delta)\big(6+\underbrace{\delta 2+\dots+\delta^k 2}_{\text{for } k \text{ periods}}+\delta^{k+1}5+\delta^{k+2}5+\dots\big)\\
&= (1-\delta)\left(6+\delta\sum_{t=0}^{k-1}\delta^t 2+\delta^{k+1}\sum_{t=0}^{\infty}\delta^t 5\right)\\
&= (1-\delta)\left(6+\delta\frac{1-\delta^k}{1-\delta}2+\delta^{k+1}\frac{5}{1-\delta}\right)\\
&= 6-4\delta+3\delta^{k+1}.
\end{aligned}$$

This one-shot deviation is unprofitable as long as $5 \geq 6-4\delta+3\delta^{k+1}$ or $3\delta^{k+1}-4\delta+1 \leq 0$, which is true for sufficiently high values of δ:

Claim 1. *There is $\delta^* \in (0,1)$, such that for all $\delta > \delta^*$ we have $3\delta^{k+1}-4\delta+1 \leq 0$.*

Proof. Define $f(\delta) \equiv 3\delta^{k+1}-4\delta+1$. Observe that $f(1) = 3\cdot 1^{k+1}-4+1 = 0$. Observe that $f'(\delta) = 3(k+1)\delta^k-4$, and $f(\cdot)$ is strictly increasing for all $\delta > \big(\frac{4}{3(k+1)}\big)^{\frac{1}{k}} \equiv \delta^*$. Note $\delta^* < 1$ since $4 < 3(1+1) \leq 3(k+1)$, hence $f(\delta) < 0$ for all $\delta \in (\delta^*, 1)$. ☐

One-shot deviations in the punishment phase are clearly unprofitable. ☐